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Review

Lie groupoids and Lie algebroids in physics and noncommutative geometry

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Abstract

Groupoids generalize groups, spaces, group actions, and equivalence relations. This last aspect dominates in noncommutative geometry, where groupoids provide the basic tool to desingularize pathological quotient spaces. In physics, however, the main role of groupoids is to provide a unified description of internal and external symmetries. What is shared by noncommutative geometry and physics is the importance of Connes's idea of associating a C^*-algebra $C^*(\Gamma)$ to a Lie groupoid Γ: in noncommutative geometry $C^*(\Gamma)$ replaces a given singular quotient space by an appropriate noncommutative space, whereas in physics it gives the algebra of observables of a quantum system whose symmetries are encoded by Γ. Moreover, Connes's map $\Gamma \mapsto C^*(\Gamma)$ has a classical analogue $\Gamma \mapsto A^*(\Gamma)$ in symplectic geometry due to Weinstein, which defines the Poisson manifold of the corresponding classical system as the dual of the so-called Lie algebroid $A(\Gamma)$ of the Lie groupoid Γ, an object generalizing both Lie algebras and tangent bundles.

Only a handful of physicists appear to be familiar with Lie groupoids and Lie algebroids, whereas the latter are practically unknown even to mathematicians working in noncommutative geometry: so much the worse for its relationship with symplectic geometry! Thus the aim of this review paper is to explain the relevance of both objects to both audiences. We do so by outlining the road from canonical quantization to Lie groupoids and Lie algebroids via Mackey's imprimitivity theorem and its symplectic counterpart. This will also lead the reader into symplectic groupoids, which define a 'classical' category on which quantization may speculatively be defined as a functor into the category $\mathfrak{KK}$ defined by Kasparov's bivariant K-theory of C^*-algebras. This functor unifies deformation quantization and geometric quantization, the conjectural functoriality of quantization counting the

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"quantization commutes with reduction" conjecture of Guillemin and Sternberg among its many consequences.

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Contents

1. Introduction

Influenced by mathematicians such as Grothendieck, Mackey, Connes, and Weinstein, the use of groupoids in pure mathematics has become respectable (though by no means widespread), at least in their respective areas of algebraic geometry, representation theory, noncommutative geometry, and symplectic geometry.[1] Unfortunately, in physics groupoids remain virtually unknown.[2] This is a pity for at least two reasons. Firstly, much of the spectacular mathematics developed in the areas just mentioned becomes inaccessible to physicists, despite its undeniable relevance to physics. This obstructs, for example, the development of a good theory for quantizing singular spaces (of the kind necessary for quantum cosmology); cf. [55]. As a case in point, many completely natural constructions in noncommutative geometry look mysterious to physicists who are not familiar with groupoids. Secondly, in the smooth setting, Lie groupoids along with their associated infinitesimal

[1] There is a Groupoid Home Page at http://www.unr.edu/homepage/ramazan/groupoid/. See also http://www.cameron.edu/∼koty/groupoids/for an incomplete but useful list of papers involving groupoids, necessarily restricted to mathematics.

[2] Conferences such as *Groupoids in Analysis, Geometry, and Physics* (Boulder, 1999, see [84]) and *Groupoids and Stacks in Physics and Geometry* (Luminy, 2004) tend te be almost exlusively attended by mathematicians.

objects called Lie algebroids provide an ideal framework for practically all aspects of both classical and quantum physics that involve symmetry in one way or the other.

Indeed, whereas in the work of Grothendieck and Connes groupoids mainly occur as generalizations of equivalence relations,[3] the role of groupoids as generalized symmetries has been emphasized by Weinstein [104]: "Mathematicians tend to think of the notion of symmetry as being virtually synonymous with the theory of groups and their actions.[4] (...) In fact, though groups are indeed sufficient to characterize homogeneous structures, there are plenty of objects which exhibit what we clearly recognize as symmetry, but which admit few or no nontrivial automorphisms. It turns out that the symmetry, and hence much of the structure, of such objects can be characterized if we use groupoids and not just groups".

The aim of this paper is to (briefly) explain what Lie groupoids and Lie algebroids are, and (more extensively) to outline which role they play in physics (at least from the perspective of the author). Because of the close relationship between quantum theory and noncommutative geometry on the one hand, and classical mechanics and symplectic geometry on the other,[5] our discussion obviously relates to matters of pure mathematics as well, and here the physics perspective turns out to be quite useful in clarifying the relationship between noncommutative and symplectic geometry. This relationship is rarely studied in noncommutative geometry, which might explain the regrettable absence of the concept of a Lie algebroid from the field.[6]

With this goal in mind, one of our main points will be to show that the role of Lie groupoids on the quantum or noncommutative side is largely paralleled by the role Lie algebroids play on the classical or symplectic side. The highlight of this philosophy is undoubtedly the close analogy between Connes's map $\Gamma \mapsto C^*(\Gamma)$ in noncommutative geometry [12] and Weinstein's map $\Gamma \mapsto A^*(\Gamma)$ in symplectic geometry [16,17], notably the functoriality of both [48]. Furthermore, the transition from classical to quantum theory through deformation quantization turns out to be given precisely by the association of the C^*-algebra $C^*(\Gamma)$ to the Poisson manifold $A^*(\Gamma)$ [47,56,83]. Hence quantization is closely related to 'integration', in the sense of the association of a Lie groupoid to a Lie algebroid; see [57] for an introduction to this problem, and [18] for its solution.

We do not provide an extensive mathematical introduction to Lie groupoids and Lie algebroids, partly because we have already done so before [46], and partly because various excellent textbooks on this subject are now available [58,68,10]. Instead, we start entirely

[3] Grothendieck (to R. Brown in a letter from 1985): "The idea of making systematic use of groupoids (notably fundamental groupoids of spaces, based on a given set of base points), however evident as it may look today, is to be seen as a significant conceptual advance, which has spread into the most manifold areas of mathematics. (...) In my own work in algebraic geometry, I have made extensive use of groupoids—the first one being the theory of the passage to quotient by a 'pre-equivalence relation' (which may be viewed as being no more, no less than a groupoid in the category one is working in, the category of schemes say), which at once led me to the notion (nowadays quite popular) of the nerve of a category. The last time has been in my work on the Teichmüller tower, where working with a 'Teichmüller groupoid' (rather than a 'Teichmüller group') is a must, and part of the very crux of the matter (...)".

[4] Cf. Connes: "It is fashionable among mathematicians to despise groupoids and to consider that only groups have an authentic mathematical status, probably because of the pejorative suffix oid" [12].

[5] Throughout this paper we use the term 'symplectic geometry' so as to include Poisson geometry.

[6] Except for the work of the author, the sole exception known to him is [70].

on the physics side, with a crash course on canonical quantization and its reformulation by Mackey in terms of systems of imprimitivity. In its original setting Mackey's notion of quantization was not only limited to homogeneous configuration spaces, but in addition lacked an underlying classical theory.[7] Both drawbacks are entirely removed once one adopts the perspective of Lie groupoids on the quantum side and Lie algebroids on the classical side, and we propose this as a convenient point of entry for physicists into the world of these seemingly strange and unfamiliar objects.

Once this perspective has been adopted, the entire theory of canonical quantization and its (finite-dimensional) generalizations is absorbed into a single theorem, stating that the association of $C^*(\Gamma)$ to $A^*(\Gamma)$ mentioned above is a 'strict' deformation quantization (in the sense of Rieffel [89,90]). Furthermore, in our opinion the deepest understanding of Mackey's imprimitivity theorem comes from its derivation from the functoriality of Connes's map $\Gamma \mapsto C^*(\Gamma)$; similarly, the classical analogue of the imprimitivity theorem in symplectic geometry [108] can be derived from the functoriality of Weinstein's map $\Gamma \mapsto A^*(\Gamma)$ already mentioned.

We finally combine the toolkit of noncommutative geometry with that of symplectic geometry in proposing a functorial approach to quantization, which is based on KK-theory on the quantum side and on symplectic groupoids on the classical side. As we see it, this approach provides the ultimate generalization of the 'quantization commutes with reduction' philosophy of Dirac [20] (in physics) and Guillemin and Sternberg [31,33] (in mathematics). Beside the use of the K-theory of C^*-algebras, this generalization hinges on the use of Lie groupoids and Lie algebroids, and therefore appears to be an appropriate endpoint of this paper.

2. From canonical quantization to systems of imprimitivity

Quantum mechanics was born in 1925 with the work of Heisenberg, who discovered the noncommutative structure of its algebra of observables [36]. The complementary work of Schrödinger from 1926 [92], on the other hand, rather started from the classical geometric structure of configuration space. Within a year, their work was unified by von Neumann, who introduced the abstract concept of a Hilbert space, in which Schrödinger's wave functions are vectors, and Heisenberg's observables are linear operators; see [72]. As every physicist knows, the basic link between matrix mechanics and wave mechanics lies in the identification of Heisenberg's infinite matrices p_j and q^i $(i, j = 1, 2, 3)$, representing the momentum and position of a particle moving in $\mathbb{R}^3$, with Schrödinger's operators $-i\hbar\partial/\partial x^j$ and x^i (seen as a multiplication operator) on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$, respectively. The key to this identification lies in the canonical commutation relations

$$[p_i, q^j] = -i\hbar\delta_i^j. \tag{1}$$

Although a mathematically rigorous theory of these commutation relations (as they stand) exists [42,91], they are problematic nonetheless. Firstly, the operators involved are unbounded, and in order to represent physical observables they have to be self-adjoint;

[7] More precisely, the underlying classical theory was not correctly identified [62].

yet on their respective domains of self-adjointness the commutator on the left-hand side is undefined. Secondly, (1) relies on the possibility of choosing global coordinates on $\mathbb{R}^3$, which precludes at least a naive generalization to arbitrary configuration spaces.[8]

Finding an appropriate mathematical interpretation of the canonical commutation relations (1) is the subject of quantization theory; see [2,46] for recent reviews. From the numerous ways to handle the situation, we here select Mackey's approach [60,61].[9] The essential point is to assign momentum and position a quite different role in quantum mechanics, despite the fact that in classical mechanics p and q can be interchanged by a canonical transformation.[10]

Firstly, the position operators q^j are collectively replaced by a single projection-valued measure P on $\mathbb{R}^3$,[11] which is given by $P_E = \chi_E$ as a multiplication operator on $L^2(\mathbb{R}^3)$. Given this P, any multiplication operator f defined by a measurable function $f : \mathbb{R}^3 \to \mathbb{R}$ can be represented as $f = \int_{\mathbb{R}^3} \mathrm{d}P_E(x)\, f(x)$, which is defined and self-adjoint on a suitable domain.[12] In particular, the position operators q^i can be reconstructed from P by choosing $f(x) = x^i$.

Secondly, the momentum operators p_i are collectively replaced by a single unitary group representation $U(\mathbb{R}^3)$ on $L^2(\mathbb{R}^3)$, defined by

$$U(y)\psi(x) := \psi(x - y).$$

Each p_i can be reconstructed from U by means of

$$p_i\psi := i\hbar \lim_{t_i \to 0} t_i^{-1}(U(t_i) - 1)\psi,$$

where $U(t_i)$ is U at $x^i = t_i$ and $x^j = 0$ for $j \neq i$; this operator is defined and self-adjoint on the set of all $\psi \in H$ for which the limit exists (Stone's theorem [79]).

[8] Mackey [[61], p. 283]: "Simple and elegant as this model is, it appears at first sight to be quite arbitrary and ad hoc. It is difficult to understand how anyone could have guessed it and by no means obvious how to modify it to fit a model for space different from $\mathbb{R}^r$".

[9] Continuing the previous quote, Mackey claims with some justification that his approach "(a) Removes much of the mystery. (b) Generalizes in a straightforward way to any model for space with a separable locally compact group of isometries. (c) Relates in an extremely intimate way to [the theory of induced representations]." In any case, Mackey's approach to the canonical commutation relations, especially in its C^*-algebraic reformulation presented below, is vastly superior to their equally C^*-algebraic reformulation in terms of the so-called Weyl C^*-algebra (cf. e.g. [8]). Indeed (see [46] Def. IV.3.5.1), the Weyl algebra over a Hilbert space $\mathcal{H}$ (which in the case at hand is $\mathbb{C}^3$) may be seen as the twisted group C^*-algebra over $\mathcal{H}$ as an abelian group under addition, *equipped with the discrete topology*. This rape of $\mathcal{H}$ as a topological space is so ugly that it is surprising that papers on the Weyl C^*-algebra continue to appear. Historically, Weyl's exponentiation of the canonical commutation relations was just one of the first attempts to reformulate a problem involving unbounded operators in terms of bounded ones, and has now been superseded.

[10] This feature is shared by most approaches to quantization, except the one mentioned in the preceding footnote.

[11] A projection-valued measure P on a space Ω with Borel structure (i.e. equipped with a σ-algebra of measurable sets defined by the topology) with values in a Hilbert space $\mathcal{H}$ is a map $E \mapsto P_E$ from the Borel subsets $E \subset \Omega$ to the projections on H that satisfies $P_\emptyset = 0$, $P_\Omega = 1$, $P_E P_F = P_F P_E = P_{E\cap F}$ for all measurable $E, F \subset \Omega$, and $P_{\cup_{i=1}^\infty E_i} = \sum_{i=1}^\infty P_{E_i}$ for all countable collections of mutually disjoint $E_i \subset \Omega$.

[12] This domain consists of all $\psi \in H$ for which $\int_{\mathbb{R}^3} \mathrm{d}(\psi, P_E(x)\psi)\, |f(x)|^2 < \infty$.

Consequently, it entails no loss of generality to work with the pair (P, U) instead of the (q^j, p_i). The commutation relations (1) are now replaced by

$$U(x)P_E U(x)^{-1} = P_{xE}, \tag{2}$$

where E is a Borel subset of $\mathbb{R}^3$ and $xE = \{x\omega \mid \omega \in E\}$. On the basis of this reformulation, Mackey proposed the following sweeping generalization of the the canonical commutation relations.[13]

Definition 1. Suppose a Lie group G acts smoothly on a manifold M.

(1) A system of imprimitivity $(\mathcal{H}, U, P)$ for this action consists of a Hilbert space $\mathcal{H}$, a unitary representation U of G on $\mathcal{H}$, and a projection-valued measure $E \mapsto P_E$ on M with values in $\mathcal{H}$, such that (2) holds for all $x \in G$ and all Borel sets $E \subset M$.
(2) A G-covariant representation $(\mathcal{H}, U, \pi)$ of the C^*-algebra $C_0(M)$ relative to this action consists of a Hilbert space $\mathcal{H}$, a unitary representation U of G on $\mathcal{H}$, and a nondegenerate representation π of $C_0(M)$ on $\mathcal{H}$ satisfying

$$U(x)\pi(\varphi)U(x)^{-1} = \pi(L_x\varphi) \tag{3}$$

for all $x \in G$ and $\varphi \in C_0(M)$, where $L_x\varphi(m) = \varphi(x^{-1}m)$.

The spectral theorem (cf. [79]) implies that these notions are equivalent: a projection-valued measure P defines and is defined by a nondegenerate representation π of $C_0(M)$ on $\mathcal{H}$ by means of $\pi(\varphi) = \int_M \mathrm{d}P(m)\varphi(m)$, and (2) is then equivalent to the covariance condition (3). Hence we may interchangeably speak of systems of imprimitivity or covariant representations. As a further reformulation, it is easy to show (cf. [21,23,78]) that there is a bijective correspondence between G-covariant representations of $C_0(M)$ and nondegenerate representations of the so-called transformation group C^*-algebra $C^*(G, M) \equiv G \times_\alpha C_0(M)$ defined by the given G-action on M, which determines an automorphic action α of G on $C_0(M)$ by $\alpha_x = L_x$.[14]

Such a system describes the quantum mechanics of a particle moving on a configuration space M on which G acts by symmetry transformations; in particular, each element X of the Lie algebra $\mathfrak{g}$ of G defines a generalized momentum operator

$$\hat{X} = i\hbar dU(X) \tag{4}$$

on $\mathcal{H}$, which is defined and self-adjoint on the domain of vectors $\psi \in \mathcal{H}$ for which

$$dU(X)\psi := \lim_{t\to 0} t^{-1}(U(\exp(tX)) - 1)\psi$$

[13] In order to maintain the connection with the classical theory later on, we restrict ourselves to Lie groups acting smoothly on manifolds. Mackey actually formulated his results more generally in terms of separable locally compact groups acting continuously on locally compact spaces.

[14] In one direction, this correspondence is as follows: given a G-covariant representation $(\mathcal{H}, \pi, U)$, one defines a representation $\pi_U(C^*(G, M))$ by extension of $\pi_U(f) = \int_G \mathrm{d}x\, \pi(f(x, \cdot))U(x)$, where $f \in C_c^\infty(G \times M) \subset C^*(G, M)$, and $f(x, \cdot)$ is seen as an element of $C_0(M)$.

exists. These operators satisfy the generalized canonical commutation relations

$$[\hat{X}, \hat{Y}] = i\hbar\widehat{[X, Y]} \tag{5}$$

and

$$[\hat{X}, \pi(\varphi)] = \pi(\xi_X \varphi), \tag{6}$$

where $\varphi \in C_c^\infty(M)$ and ξ_X is the canonical vector field on M defined by the G-action; of course, these should be supplemented with

$$[\pi(\varphi_1), \pi(\varphi_2)] = 0. \tag{7}$$

Elementary quantum mechanics on $\mathbb{R}^n$ then corresponds to the special case $M = \mathbb{R}^n$ and $G = \mathbb{R}^n$ with the usual additive group structure.

3. The imprimitivity theorem

In the spirit of the C^*-algebraic approach to quantum physics [81,96,24,34], the C^*-algebra $C^*(G, M)$ defined by the given G-action on M should be seen as an algebra of observables, whose inequivalent irreducible representations define the possible superselection sectors of the system. As we have seen, these representation s may equivalently be seen as systems of imprimitivity or as G-covariant representation s of $C_0(M)$ [21,23,78]. In any case, it is of some interest to classify these. Mackey's *imprimitivity theorem* describes the simplest case where this is possible.

Theorem 1. [7,59] *Let H be a closed subgroup of G and let G act on $M = G/H$ by left translation. Up to unitary equivalence, there is a bijective correspondence between systems of imprimitivity $(\mathcal{H}, U, P)$ for this action (or, equivalently, G-covariant representation of $C_0(G/H)$ or nondegenerate representations of the transformation group C^*-algebra $C^*(G, G/H)$) and unitary representations U_χ of H, as follows*:

- *Given $U_\chi(H)$ on a Hilbert space $\mathcal{H}_\chi$, the triple $(\mathcal{H}^\chi, U^\chi, P^\chi)$ is a system of imprimitivity, where $\mathcal{H}^\chi = L^2(G/H, G \times_H \mathcal{H}_\chi)$ is the Hilbert space of L^2-sections of the vector bundle $G \times_H \mathcal{H}_\chi$ associated to the principal H-bundle G over G/H by U_χ, U^χ is the representation of G induced by U_χ, and $P^\chi_E = \chi_E$ acts canonically on $\mathcal{H}^\chi$ as a multiplication operator.*
- *Conversely, if $(\mathcal{H}, U, P)$ is a system of imprimitivity, then there exists a unitary representation $U_\chi(H)$ such that the triple $(\mathcal{H}, U, P)$ is unitarily equivalent to the triple $(\mathcal{H}^\chi, U^\chi, P^\chi)$ just described.*

The correspondence $(\mathcal{H}_\chi, U_\chi) \leftrightarrow (\mathcal{H}^\chi, U^\chi, P^\chi)$ preserves direct sums and, accordingly, irreducibility.

The simplest and at the same time most beautiful application of the imprimitivity theorem is Mackey's recovery of the Stone–von Neumann uniqueness theorem concerning the (regular) irreducible representations of the canonical commutation relations: taking $G = \mathbb{R}^3$ and $H = \{e\}$ (so that $M = \mathbb{R}^3$), one finds that the associated system of imprimitivity possesses

precisely one irreducible representation, since the trivial group obviously has only one such representation.[15] Furthermore (and this was one of Mackey's main points), one may keep $\mathbb{R}^3$ as a confuguration space but replace $G = \mathbb{R}^3$ by the Euclidean group $G = SO(3) \ltimes \mathbb{R}^3$, so that $H = SO(3)$. The generalized momenta then include the angular momentum operators J^i along with their commutation relations, and the imprimitivity theorem then asserts that the irreducible representations of (2) correspond to the usual irreducible representations U_j of $SO(3)$, $j = 0, 1, \ldots$.[16] Mackey saw this as an explanation for the emergence of spin as a purely quantum-mechanical degree of freedom; the latter perspective of spin goes back to the pionieers of quantum theory [77], but is now obsolete (see Section 9 below).

Mackey's imprimitivity theorem admits a generalization to G-actions on an arbitrary manifold M, provided the action is regular.[17]

Proposition 1. [26,27] *Suppose that each G-orbit in M is (relatively) open in its closure. The irreducible representations of $C^*(G \ltimes M)$ are classified by pairs $(\mathcal{O}, U_\chi)$, where $\mathcal{O}$ is a G-orbit in M and U_χ is an irreducible representation of the stabilizer of an arbitrary point $m_0 \in \mathcal{O}$.*[18]

In view of the power of Mackey's imprimitivity theorem, both for representation theory and quantization theory, increasingly sophisticated and insightful proofs have been published over the last five decades.[19] All proofs relevant to noncommutative geometry are either based on or are equivalent to:

Theorem 2. [30,87] *The transformation group C^*-algebra $C^*(G, G/H)$ is Morita equivalent to $C^*(H)$.*

This means that there exists a so-called equivalence or imprimitivity bimodule $\mathcal{E}$ (which in modern terms would be called a $C^*(G, G/H)$-$C^*(H)$ Hilbert bimodule)[20] that allows one to set up the bijective correspondence – called for in Mackey's imprimitivity theorem – between (nondegerenerate) representation s of $C^*(G, G/H)$ and those of $C^*(H)$ (or

[15] The "uniqueness of the canonical commutation relations" has also been derived from the fact that (up to unitary equivalence) there is only one irreducible representation of any of the following objects: (i) The Heisenberg Lie group with given nonzero central charge (von Neumann's theorem [71]); (ii) The Weyl C^*-algebra over a finite-dimensional Hilbert space, provided one restricts oneself to the class of regular representations [8]; or (iii) The C^*-algebra of compact operators [86].

[16] By the usual arguments, one may replace $SO(3)$ by $SU(2)$ in this argument, so as to obtain $j = 0, 1/2, \ldots$.

[17] In view of this simple result, C^*-algebra ists are mainly interested in nonregular actions, cf. [23], but for physics Proposition 1 is quite useful. In any case, an example of a nonregular action is the action of $\mathbb{Z}$ on $\mathbb{T}$ by irrational rotations.

[18] The associated G-covariant representation of $C_0(M)$ may be realized by multiplication operators on the Hilbert space $\mathcal{H}^\chi$ carrying the representation $U^\chi(G)$ induced by U_χ.

[19] Mackey's own proof was rather measure-theoretic in flavour, and did not shed much light on the origin of his result. Probably the shortest proof is [74].

[20] A Hilbert bimodule $A \circlearrowleft \mathcal{E} \circlearrowright B$ over C^*-algebras A and B consists of a Banach space $\mathcal{E}$ that is an algebraic A-B bimodule, and is equipped with a B-valued inner product that is compatible with the A and B actions. Such objects were first considered by Rieffel [87], who defined an 'interior' tensor product $\mathcal{E}\hat{\otimes}_B\mathcal{F}$ of an A–B Hilbert bimodule $\mathcal{E}$ with a B–C Hilbert bimodule $\mathcal{F}$, which is an A–C Hilbert bimodule.

equivalently, of H). Given a unitary representation $U_\chi(H)$ on a Hilbert space $\mathcal{H}_\chi$, or the associated representation π_χ of $C^*(H)$ on the same space, one constructs a Hilbert space $\mathcal{H}^\chi = \mathcal{E}\hat{\otimes}_{\pi_\chi}\mathcal{H}_\chi$. The action of $C^*(G, G/H)$ on $\mathcal{E}$ descends to an action $\pi^\chi(C^*(G, G/H))$ on $\mathcal{H}^\chi$, and extracting the associated representation s of G and of $C_0(G/H)$ one finds that this is precisely Mackey's induction construction paraphrased in Theorem 1. Conversely, a given representation π^χ of $C^*(G, G/H)$ on a Hilbert space $\mathcal{H}^\chi$ defines $\pi_\chi(C^*(H))$ on $\mathcal{H}_\chi = \bar{\mathcal{E}}\hat{\otimes}_{\pi^\chi}\mathcal{H}^\chi$, and this process is the inverse of the previous one. Replacing the usual algebraic bimodule tensor product by Rieffel's interior tensor product $\hat{\otimes}_\pi$, this entirely mimics the corresponding procedure in algebra (cf. [25]); in the same spirit, one infers also in general that two Morita equivalent C^*-algebras have equivalent representation categories.

The reformulation of Theorem 1 as Theorem 2 begs the question what the deeper origin of the latter could possibly be. One answer is given by the analysis in [22], from which Theorem 2 emerges as merely a droplet in an ocean of imprimitivity theorems. The answer below [48–50,69] is equally categorical in spirit, but is entirely based on the use of Lie groupoids. Namely, we will derive Theorem 2 and hence Mackey's Theorem 1 from the functoriality of Connes's map (8) below, which associates a C^*-algebra to a Lie groupoid. Apart from the fact that this is very much in the spirit of noncommutative geometry, the use of Lie groupoids will enable us to formulate an analogous classical procedure in terms of Lie algebroids and Poisson manifolds. All this requires a little preparation.

4. Intermezzo: Lie groupoids

Recall that a *groupoid* is a small category (i.e. a category in which the underlying classes are sets) in which each arrow is invertible. We denote the total space (i.e. the set of arrows) of a groupoid Γ by Γ_1, and the base space (i.e. the set on which the arrows act) by Γ_0; the object inclusion map $\Gamma_0 \hookrightarrow \Gamma_1$ is written $u \mapsto 1_u$. We denote the inverse $\Gamma_1 \to \Gamma_1$ by $x \mapsto x^{-1}$, and the source and target maps by $s, t : \Gamma_1 \to \Gamma_0$. Thus the composable pairs form the space $\Gamma_2 := \{(x, y) \in \Gamma_1 \times \Gamma_1 \mid s(x) = t(y)\}$, so that if $(x, y) \in \Gamma_2$ then $xy \in \Gamma_1$ is defined.[21] A *Lie groupoid* is a groupoid for which Γ_1 and Γ_0 are manifolds (Γ_1 not necessarily being Hausdorff), s and t are surjective submersions, and multiplication and inversion are smooth.[22] See [58,68] for recent textbooks on Lie groupoids and related matters.[23]

Some examples of Lie groupoids that are useful to keep in mind are:

- A *Lie group* G, where $\Gamma_1 = G$ and $\Gamma_0 = \{e\}$).
- A *manifold* M, where $\Gamma_1 = \Gamma_0 = M$ with the obvious trivial groupoid structure $s(x) = t(x) = 1_x = x^{-1} = x$, and $xx = x$.
- The *pair groupoid* over a manifold M, where $\Gamma_1 = M \times M$ and $\Gamma_0 = M$, with $s(x, y) = y$, $t(x, y) = x$, $(x, y)^{-1} = (y, x)$, $(x, y)(y, z) = (x, z)$, and $1_x = (x, x)$.

[21] Thus the axioms are: 1. $s(xy) = s(y)$ and $t(xy) = t(x)$; 2. $(xy)z = x(yz)$ 3. $s(1_u) = t(1_u) = u$ for all $u \in \Gamma_0$; 4. $x1_{s(x)} = 1_{t(x)}x = x$ for all $x \in \Gamma_1$.

[22] It follows that object inclusion is an immersion, that inversion is a diffeomorphism, that Γ_2 is a closed submanifold of $\Gamma_1 \times \Gamma_1$, and that for each $u \in \Gamma_0$ the fibers $s^{-1}(u)$ and $t^{-1}(u)$ are submanifolds of Γ_1.

[23] The concept of a Lie groupoid was introduced by Ehresmann.

- The *gauge groupoid* defined by a principal H-bundle $P \xrightarrow{\pi} M$, where $\Gamma_1 = P \times_H P$ (which stands for $(P\times P)/H$ with respect to the diagonal H-action on $P\times P$), $\Gamma_0 = M$, $s([p,q]) = \pi(q)$, $t([p,q]) = \pi(p)$, $[x,y]^{-1} = [y,x]$, and $[p,q][q,r] = [p,r]$ (here $[p,q][q',r]$ is defined whenever $\pi(q) = \pi(q')$, but to write down the product one picks $q \in \pi^{-1}(q')$).
- The *action groupoid* $G \ltimes M$ defined by a smooth (left) action $G \circlearrowright M$ of a Lie group G on a manifold M, where $\Gamma_1 = G\times M$, $\Gamma_0 = M$, $s(g,m) = g^{-1}m$, $t(g,m) = m$, $(g,m)^{-1} = (g^{-1}, g^{-1}m)$, and $(g,m)(h,g^{-1}m) = (gh,m)$.

As mentioned before, an equivalence relation on a set M defines a groupoid, namely the obvious subgroupoid of the pair groupoid over M. However, in interesting examples this is rarely a Lie groupoid. To obtain a Lie groupoid resembling a given equivalence relation on a manifold, various refinements of the subgroupoid in question have been invented, of which the holonomy groupoid defined by a foliation is the most important example for noncommutative geometry [12,68,80].

For reasons to emerge from the ensuing story, we look at Lie groupoids as objects in the *category of principal bibundles*. To define this category, we first recall that an action of a groupoid Γ on a space M is only defined if M comes equipped with a map $M \xrightarrow{\pi} \Gamma_0$. In that case, a left Γ action on M is a map $(x,m) \mapsto xm$ from $\Gamma_1 \times^{s,\pi}_{\Gamma_0} M$ to M,[24] such that $\pi(xm) = t(x)$, $xm = m$ for all $x \in \Gamma_0$, and $x(ym) = (xy)m$ whenever $s(y) = \tau(m)$ and $t(y) = s(x)$. Similarly, given a map $M \xrightarrow{\rho} \Delta_0$, a right action of a groupoid Δ on M is a map $(m,h) \mapsto mh$ from $M \times^{\rho,t}_{\Delta_0} \Delta_1$ to M that satisfies $\rho(mh) = s(h)$, $mh = m$ for all $h \in \Delta_0$, and $(mh)k = m(hk)$ whenever $\rho(m) = t(h)$ and $t(k) = s(h)$. Now, if Γ and Δ are groupoids, a Γ–Δ bibundle M, also written as $\Gamma \circlearrowright M \circlearrowleft \Delta$, carries a left Γ action as well as a right Δ-action that commute.[25] Such a bibundle is called *principal* when $\pi : M \to \Gamma_0$ is surjective, and the Δ action is free (in that $mh = m$ iff $h \in \Delta_0$) and transitive along the fibers of π.

Suppose one has right principal bibundles $\Gamma \circlearrowright M \circlearrowleft \Delta$ and $\Delta \circlearrowright N \circlearrowleft \Theta$. The fiber product $M \times_{\Delta_0} N$ carries a right Δ action, given by $h : (m,n) \mapsto (mh, h^{-1}n)$ (defined as appropriate). The orbit space $(M \times_{\Delta_0} N)/\Delta$ is a $\Gamma - \Theta$ bibundle in the obvious way inherited from the original actions. Thus, regarding $\Gamma \circlearrowright M \circlearrowleft \Delta$ as an arrow from Γ to Δ and $\Delta \circlearrowright N \circlearrowleft \Theta$ as an arrow from Δ to Θ, one map look upon $\Gamma \circlearrowright (M \times_\Delta N)/\Delta \circlearrowleft \Theta$ as an arrow from Γ to Θ, defining the product or composition of M and N. However, this product is associative merely up to isomorphism, so that in order to have a category one should regard isomorphism classes of principal bibundles as arrows.

For Lie groupoids everything in these definitions has to be smooth (and π a surjective submersion).

Definition 2. [11,35,38,67,68] The category $\mathfrak{G}$ of Lie groupoids and principal bibundles has Lie groupoids as objects and isomorphism classes $[\Gamma \circlearrowright M \circlearrowleft \Delta]$ of principal bibundles

[24] Here we use the notation $A \times^{f,g}_B C = \{(a,c) \in A\times C \mid f(a) = g(c)\}$ for the fiber product of sets A and C with respect to maps $f : A \to B$ and $g : C \to B$.

[25] That is, one has $\tau(mh) = \tau(m)$, $\rho(xm) = \rho(m)$, and $(xm)h = x(mh)$ whenever defined.

as arrows. Composition of arrows is given by

$$[\Gamma \circlearrowleft M \circlearrowright \Delta] \circ [\Delta \circlearrowleft N \circlearrowright \Theta] = [\Gamma \circlearrowleft (M \times_\Delta N)/\Delta \circlearrowright \Theta],$$

and the identities are given by $1_\Gamma = [\Gamma \circlearrowleft \Gamma \circlearrowright \Gamma]$, seen as a bibundle in the obvious way.

Of course, it can be checked that this definition is correct in the sense that one indeed defines a category in this way. This category has the remarkable feature that (Morita) equivalence of groupoids (as defined in [69], a notion heavily used in noncommutative geometry) is the same as isomorphism of objects in $\mathfrak{G}$.

5. From Lie groupoids to the imprimitivity theorem

A central idea in noncommutative geometry is the association

$$\Gamma \mapsto C^*(\Gamma) \tag{8}$$

of a C^*-algebra $C^*(\Gamma)$ to a Lie groupoid Γ [12].[26] Here $C^*(\Gamma)$ is a suitable completion of the function space $C_c^\infty(\Gamma_1)$, equipped with a convolution-type product defined by the groupoid structue. For the above examples, this yields:

- The C^*-algebra of a Lie group G is the usual convolution C^*-algebra $C^*(G)$ defined by the Haar measure on G [78].
- For a manifold M one has $C^*(M) = C_0(M)$.
- The pair groupoid over a connected manifold M defines $C^*(M \times M) \cong K(L^2(M))$, i.e. the C^*-algebra of compact operators on the L^2-space canonically defined by a manifold.
- The C^*-algebra defined by a gauge groupoid $P \times_H P$ as above is isomorphic to $K(L^2(M)) \otimes C^*(H)$ (but any explicit isomorphism depends on the choice of a measurable section $s : M \to P$, which in general cannot be smooth).
- For an action groupoid defined by $G \circlearrowleft M$ one has $C^*(G \ltimes M) \cong C^*(G, M)$, the transformation group C^*-algebra defined by the given action [23,78].

Having already defined the category $\mathfrak{G}$ of principal bibundles for Lie groupoids, in order to make the map (8) functorial, one has to regard C^*-algebras as objects in a suitable category $\mathfrak{C}$ as well.

Definition 3. [22,49,93] The category $\mathfrak{C}$ has C^*-algebras as objects and isomorphism classes $[A \circlearrowleft \mathcal{E} \circlearrowright B]$ of Hilbert bimodules, as arrows, composed using Rieffel's interior tensor product. The identities are given by $1_A = A \circlearrowleft A \circlearrowright A$, defined in the obvious way.

[26] See also [46,56,76] for detailed presentations. For a Lie groupoid Γ Connes's $C^*(\Gamma)$ is the same (up to isomorphism of C^*-algebras) as the C^*-algebra Renault associates to a locally compact groupoid with Haar system [85], provided one takes the Haar system canonically defined by the smooth structure on Γ.

A crucial feature of this construction is that the notion of isomorphism of objects in $\mathfrak{C}$ coincides with Rieffel's (strong) Morita equivalence of C^*-algebras.

Theorem 3. [48] *Connes's map $\Gamma \mapsto C^*(\Gamma)$ is functorial from the category $\mathfrak{G}$ of Lie groupoids and principal bibundles to the category $\mathfrak{C}$ of C^*-algebras and Hilbert bimodules.*

Corollary 1. [50,69] *Connes's map $\Gamma \mapsto C^*(\Gamma)$ preserves Morita equivalence, in the sense that if Γ and Δ are Morita equivalent Lie groupoids, then $C^*(\Gamma)$ and $C^*(\Delta)$ are Morita equivalent C^*-algebras.*

The imprimitivity bimodule $C^*(\Gamma) \circlearrowleft \mathcal{E} \circlearrowright C^*(\Delta)$ establishing the Morita equivalence of $C^*(\Gamma)$ and $C^*(\Delta)$ is obtained from the principal bibundle $\Gamma \circlearrowleft E \circlearrowright \Delta$ establishing the Morita equivalence of Γ and Δ in a very simple way, amounting to the completion of $C_c^\infty(\Gamma) \circlearrowleft C_c^\infty(E) \circlearrowright C_c^\infty(\Delta)$; see [48,95].

For example, in Mackey's case one has $\Gamma = G \ltimes (G/H)$ and $\Delta = H$, linked by the principal bibundle $G \ltimes (G/H) \circlearrowleft G \circlearrowright H$ in the obvious way;[27] the associated imprimitivity bimodule for $C^*(G \ltimes (G/H)) \cong C^*(G, G/H)$ and $C^*(H)$ is precisely the one found by Rieffel [87]. Thus Theorem 2, and thereby Mackey's imprimitivity theorem, ultimately derives from the Morita equivalence

$$G \ltimes (G/H) \sim H \tag{9}$$

of groupoids, which is an almost trivial fact once the appropriate framework has been set up. This framework cannot be specified in terms of groups and group actions alone, despite the fact that the two groupoids relevant to Mackey's imprimitivity theorem reduce to those.

Mackey's analysis of the canonical commutation relations admits various other generalizations than Proposition 1, at least one of which is related to groupoids as well: instead of generalizing the action groupoid $G \ltimes (G/H)$ to an arbitrary action groupoid $G \ltimes M$, one may note the isomorphism of groupoids

$$G \ltimes (G/H) \cong G \times_H G, \tag{10}$$

where the right-hand side is the gauge groupoid of the principal H-bundle G with respect to the natural right-action of H. This isomorphism (given by $(xy^{-1}, \pi(x)) \leftrightarrow [x, y]$) naturally passes to the 'algebra of observables,' i.e. one has

$$C^*(G \ltimes (G/H)) \cong C^*(G \times_H G), \tag{11}$$

and one may see the right-hand side as a special case of $C^*(P \times_H P)$ for an arbitrary principal H-bundle P.[28] Here one has a complete analogue of Mackey's imprimitivity theorem: the Morita equivalence

$$P \times_H P \sim H \tag{12}$$

[27] For example, $(g_1, m)g_2 = g_1 g_2$, defined whenever $m = \pi(g_1 g_2)$.

[28] This generalization is closely related to Kaluza–Klein theory and the Wong equations; see [46].

at the groupoid level[29] induces a Morita equivalence

$$C^*(P\times_H P) \sim C^*(H) \tag{13}$$

at the C^*-algebraic level, which in turn implies that there is a bijective correspondence between (irreducible) unitary representations $U_\chi(H)$ and representations $\pi^\chi(C^*(P\times_H P))$.[30]

In the old days, the various irreducible representations (or superselection sectors) of algebras of observables like $C^*(G \ltimes M)$ or $C^*(P\times_H P)$ were seen as 'inequivalent quantizations' of a single underlying classical system. From this perspective, quantities like spin were seen as degrees of freedom peculiar to and emergent from quantum theory. Starting with geometric quantization in the mid-1960s, however, it became clear that each superselection sectors of said type is in fact the quantization of a different classical system. The language of Lie groupoids and Lie algebroids allows the most precise and conceptually clearest discussion of this situation. Mathematically, what is at stake here is the relationship between noncommutative geometry and symplectic geometry as its classical analogue.[31] We now turn to this language.

6. Intermezzo: Lie algebroids and Poisson manifolds

Since the notion of a Lie algebroid cannot found in the noncommutative geometry literature, we provide a complete definition.[32]

Definition 4. A Lie algebroid A over a manifold M is a vector bundle $A\xrightarrow{\pi} M$ equipped with a vector bundle map $A\xrightarrow{\alpha} TM$ (called the *anchor*), as well as with a Lie bracket $[\ ,\ ]$ on the space $C^\infty(M, A)$ of smooth sections of A, satisfying the Leibniz rule

$$[\sigma_1, f\sigma_2] = f[\sigma_1, \sigma_2] + (\alpha\circ\sigma_1 f)\,\sigma_2 \tag{14}$$

for all $\sigma_1, \sigma_2 \in C^\infty(M, A)$ and $f \in C^\infty(M)$.

It follows that the map $\sigma \mapsto \alpha\circ\sigma : C^\infty(M, A) \to C^\infty(M, TM)$ induced by the anchor is a homomorphism of Lie algebras, where the latter is equipped with the usual commutator of vector fields.[33]

Lie algebroids generalize (finite-dimensional) Lie algebras as well as tangent bundles, and the (infinite-dimensional) Lie algebra $C^\infty(M, A)$ could be said to be of geometric origin in the sense that it derives from an underlying finite-dimensional geometrical object. Similar

[29] The equivalence bibundle is $P\times_H P \circlearrowleft P \circlearrowright H$, with the given right H action on P and the left action given by $[x, y]y = x$.

[30] Given $U_\chi(H)$ on a Hilbert space $\mathcal{H}_\chi$, the representation π^χ is naturally realized on $L^2(P/H, P\times_H\mathcal{H}_\chi)$, as in the homogeneous case.

[31] See also [70] for a different approach to this relationship.

[32] Cf. [58,68] for detailed treatments. The concept of a Lie algebroid and the relationship between Lie groupoids and Lie algebroids are originally due to Pradines.

[33] This homomorphism property used to be part of the definition of a Lie algebroid, but as observed by Marius Crainic it follows from the stated definition.

to our list of example of Lie groupoids in Section 4, one has the following basic classes of Lie algebroids.

- A *Lie algebra* $\mathfrak{g}$, where $A = \mathfrak{g}$ and M is a point (which may be identified with the identity element of any Lie group with Lie algebra g; see below) and $\alpha = 0$.
- A *manifold* M, where $A = M$, seen as the zero-dimensional vector bundle over M, evidently with identically vanishing Lie bracket and anchor.
- The *tangent bundle* over a manifold M, where $A = TM$ and $\alpha = \mathrm{id} : TM \to TM$, with the Lie bracket given by the usual commutator of vector fields.
- The *gauge algebroid* defined by a principal H-bundle $P \to M$; here $A = (TP)/H$, so that $C^\infty(M, A) \cong C^\infty(M, TP)^H$, which inherits the commutator from $C^\infty(M, TP)$ as the Lie bracket defining the algebroidstructure, and is equipped with the projection $\alpha : (TP)/H \to TM$ induced by $TP \to TM$.
- The *action algebroid* $\mathfrak{g} \ltimes M$ defined by a $\mathfrak{g}$-action on a manifold M (i.e. a Lie algebra homomorphism $\mathfrak{g} \to C^\infty(M, TM)$) has $A = \mathfrak{g} \times M$ (as a trivial bundle) and $\alpha(X, m) = -\xi_X(m) \in T_m M$. The Lie bracket is

$$[X, Y](m) = [X(m), Y(m)]_{\mathfrak{g}} + \xi_Y X(m) - \xi_X Y(m).$$

It is no accident that these examples exactly correspond to our previous list of Lie groupoids: as for groups, any Lie groupoid Γ has an associated Lie algebroid $A(\Gamma)$ with the same base space.[34] Namely, as a vector bundle $A(\Gamma)$ is the restriction of $\ker(t_*)$ to Γ_0, and the anchor is $\alpha = s_*$. One may identify sections of $A(\Gamma)$ with left-invariant vector fields on Γ, and under this identification the Lie bracket on $C^\infty(\Gamma_0, A(\Gamma))$ is by definition the commutator.

Conversely, one may ask whether a given Lie algebroid A is *integrable*, in that it comes from a Lie groupoid Γ in the said way. That is, is $A \cong A(\Gamma)$ for some Lie groupoid Γ? This is not necessarily the case; see [18,57].

The modern interplay between Lie Lie groupoids and Lie algebroids on the ond hand, and symplectic geometry on the other is based on various amazing points of contact. The simplest of these is as follows.

Proposition 2. [16,17] *The dual vector bundle A^* of a Lie algebroid A is canonically a Poisson manifold. The Poisson bracket on $C^\infty(A^*)$ is defined by the following special cases*: $\{f, g\}_\pm = 0$ *for* $f, g \in C^\infty(M)$; $\{\tilde{\sigma}, f\} = \alpha \circ \sigma f$, *where* $\tilde{\sigma} \in \widetilde{C^\infty(A^*)}$ *is defined by a section* σ *of* A *through the obvious pairing, and finally* $\{\tilde{\sigma}_1, \tilde{\sigma}_2\} = \widetilde{[\sigma_1, \sigma_2]}$.

Conversely, if a vector bundle $E \to M$ is a Poisson manifold such that the Poisson bracket of two linear functions is linear, then $E \cong A^*$ for some Lie algebroid A over M, with the above Poisson structure.[35]

[34] The association $\Gamma \to A(\Gamma)$ is functorial in an appropriate way, so that Mackenzie speaks of the *Lie functor* [58].

[35] This establishes a categorical equivalence between linear Poisson structures on vector bundles and Lie algebroids. One can also show that in this situation the differential forms on A form a differential graded algebra, while those on $A^* \cong E$ (or, equivalently, the so-called polyvector fields on A) are a Gerstenhaber algebra; see [40].

The main examples are:

- The dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ acquires its canonical Lie–Poisson structure (cf. [63]).
- A manifold M, seen as the dual to the zero-dimensional vector bundle $M \to M$, carries the zero Poisson structure.
- A cotangent bundle T^*M acquires the Poisson structure defined by its standard symlectic structure.
- The dual $(T^*P)/H$ of a gauge algebroid inherits the canonical Poisson structure from T^*P under the isomorphism $C^\infty(T^*P)/H \cong C^\infty(T^*P)^H$.
- The dual $\mathfrak{g}^* \ltimes M$ of an action algebroid acquires the so-called *semidirect product* Poisson structure [45,64].[36]

Combining the associations $\Gamma \mapsto A(\Gamma)$ and $A \mapsto A^*$, one has an association

$$\Gamma \mapsto A^*(\Gamma), \tag{15}$$

of a Poisson manifold to a Lie groupoid, which we call *Weinstein's map*. As we shall see, this is a classical analogue of Connes's map (8) in every possible respect.

7. Symplectic groupoids and the category of Poisson manifolds

Another important point of contact between Poisson manifolds and Lie algebroids that is relevant for what follows is the following construction.

Proposition 3. [16] *If P is a Poisson manifold, then T^*P is canonically a Lie algebroid over P.*

The anchor is just the usual map $T^*P \to TP$, $\alpha \mapsto \alpha^\sharp$ (e.g., $df \mapsto X_f$)[37] defined by the Poisson structure, whereas the Lie bracket is

$$[\alpha, \beta] = \mathcal{L}_{\alpha^\sharp}\beta - \mathcal{L}_{\beta^\sharp}\alpha + d\pi(\alpha, \beta), \tag{16}$$

where π is the Poisson tensor. Combining this with Proposition 2, one infers that TP is a Poisson manifold whenever P is.[38]

The following definition will play a key role for us in many ways.

Definition 5. [16] A Poisson manifold P is called integrable when the associated Lie algebroid T^*P is integrable (in being the Lie algebroid of some Lie groupoid).

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[36] Relative to a basis of $\mathfrak{g}$ with structure constants C^c_{ab}, this is given by

$\{f, g\} = C^c_{ab}\theta_c \frac{\partial f}{\partial \theta_a}\frac{\partial g}{\partial \theta_b} + \xi_a f \frac{\partial g}{\partial \theta_a} - \frac{\partial f}{\partial \theta_a}\xi_a g$.

[37] The Hamiltonian vector field X_f defined by a smooth function f on a Poisson manifold P is defined by $X_f g = \{f, g\}$.

[38] In addition, one may recover the Poisson cohomology of P as the Lie algebroid cohomology of T^*P [58,103].

If P is an integrable Poisson manifold, a groupoid $\Gamma(P)$ for which $A(\Gamma(P)) \cong T^*P$ (and hence $\Gamma(P)_0 \cong P$) turns out to have the structure of a *symplectic groupoid*.

Definition 6. [43,100,107] A symplectic groupoid is a Lie groupoid whose total space Γ_1 is a symplectic manifold, such that the graph of $\Gamma_2 \subset \Gamma \times \Gamma$ is a Lagrangian submanifold of $\Gamma \times \Gamma \times \Gamma^-$.

See also [16,58,66]. Symplectic groupoids have many amazing properties, and in our opinion their introduction into symplectic geometry has been the biggest leap forward since the subject was founded.[39] For example:

(1) There exists a unique Poisson structure on Γ_0 such that t is a Poisson map and s is an anti-Poisson map.
(2) Γ_0 is a Lagrangian submanifold of Γ_1.
(3) The inversion in Γ is an anti-Poisson map.
(4) The foliations of Γ defined by the levels of s and t are mutually symplectically orthogonal.
(5) If Γ is s-connected,[40] then $s^*C^\infty(\Gamma_0)$ and $t^*C^\infty(\Gamma_0)$ are each other's Poisson commutant.
(6) The symplectic leaves of Γ_0 are the connected components of the Γ_1-orbits.

With regard to the first point, the Poisson structure on $\Gamma(P)_0$ induces the given one on P under the diffeomorphism $\Gamma(P)_0 \cong P$. For later use, we record:

Proposition 4. [16,19,49] *If a Poisson manifold P is integrable, then there exists an s-connected and s-simply connected symplectic groupoid $\Gamma(P)$ over P, which is unique up to isomorphism.*

For example, suppose that Δ is a Lie groupoid; is the Poisson manifold $A^*(\Delta)$ it defines by (15) integrable? The answer is yes, and one may take

$$\Gamma(A^*(\Delta)) = T^*\Delta, \tag{17}$$

the so-called *cotangent groupoid* of Δ [16] (see also [50,58]). This is s-connected and s-simply connected iff Δ is.

Using the above constructions, we now define a category $\mathfrak{P}$ of Poisson manifolds, which will play a central role in what follows. First, the objects of $\mathfrak{P}$ are *integrable* Poisson manifolds; the integrability condition turns out to be necessary in order to have identities in $\mathfrak{P}$; see below. In the spirit of general Morita theory [25], the arrows in $\mathfrak{P}$ are bimodules in an appropriate sense. Bimodules for Poisson manifolds are known as *dual pairs* [44,101].

[39] It would be tempting to say that a suitable analogue of a symplectic groupoid has not been found in noncommutative geometry so far, but in fact an analysis of the categorical significance of symplectic groupoids, Poisson manifolds, and operator algebras [49] shows that the 'quantum symplectic groupoid' associated to a C^*-algebra A is just A itself, whereas for a von Neumann algebra its standard form plays this role.

[40] This means that each fiber $s^{-1}(u)$ is connected, $u \in \Gamma_0$. Similarly for s-simply connected.

A dual pair $Q \leftarrow S \rightarrow P$ consists of a symplectic manifold S, Poisson manifolds Q and P, and complete Poisson maps $q : S \rightarrow Q$ and $p : S \rightarrow P^-$, such that $\{q^* f, p^* g\} = 0$ for all $f \in C^\infty(Q)$ and $g \in C^\infty(P)$. To explain the precise class of dual pairs whose isomorphism classes form the arrows in $\mathfrak{P}$, we need a symplectic analogue $\mathfrak{S}$ of the category $\mathfrak{G}$ (cf. Definition 2). In preparation, we call an action of a symplectic groupoid Γ on a symplectic manifold S *symplectic* when the graph of the action in $\Gamma \times S \times S^-$ is Lagrangian [16,66].

Definition 7. [49] The category $\mathfrak{S}$ is the subcategory of the category $\mathfrak{G}$ (of Lie groupoids and principal bibundles) whose objects are symplectic groupoids and whose arrows are isomorphism classes of principal bibundles for which the two groupoid actions are symplectic.

We call such bibundles *symplectic*. As we have seen (cf. Section 6), the base space of a symplectic groupoid is a Poisson manifold. Moreover, it can be shown [16,66] that the base map $S \rightarrow \Gamma_0$ of a symplectic action of a symplectic groupoid Γ on a symplectic manifold S is a complete Poisson map such that for $(\gamma, y) \in \Gamma \times^{s,\rho}_{\Gamma_0} S$ with $\gamma = \varphi_1^{t^* f}(\rho(y))$, one has $\gamma y = \varphi_1^{\rho^* f}(y)$ (here φ_t^g is the Hamiltonian flow induced by a function g, and $f \in C^\infty(\Gamma_0)$). Conversely, when Γ is s-connected and s-simply connected, a given complete Poisson map $\rho : S \rightarrow \Gamma_0$ is the base map of a unique symplectic Γ action on S with the above property [105]. Furthermore, it is easy to show that the base maps of a symplectic bibundle form a dual pair. We call a dual pair arising from a symplectic principal bibundle in this way *regular*.

Definition 8. The objects of the category $\mathfrak{P}$ of Poisson manifolds and dual pairs are integrable Poisson manifolds, and its arrows are isomorphism classes of regular dual pairs.

The identities in $\mathfrak{P}$ are $1_P = [P \leftarrow \Gamma(P) \rightarrow P]$, where $\Gamma(P)$ is "the" s-connected and s-simply connected symplectic groupoid over P; cf. Proposition 4. As in every decent version of Morita theory, isomorphism of objects in $\mathfrak{P}$ comes down to Morita equivalence of Poisson manifolds (in the sense of Xu [105]).

It is clear that $\mathfrak{P}$ is equivalent to the full subcategory $\mathfrak{S}_c$ of $\mathfrak{S}$ whose objects are s-connected and s-simply connected symplectic groupoids; the advantage of working with $\mathfrak{P}$ rather than $\mathfrak{S}_c$ lies both in the greater intuitive appeal of Poisson manifolds and dual pairs over symplectic groupoids and symplectic principal bibundles, and also in the fact that the composition of arrows can be formulated in direct terms (i.e. avoiding arrow composition in $\mathfrak{S}$ or $\mathfrak{G}$) using a generalization of the familiar procedure of symplectic reduction [49,106].

For example, a strongly Hamiltonian group action $G \circlearrowleft S$ famously defines a dual pair

$$S/G \xleftarrow{\pi} S \xrightarrow{J} \mathfrak{g}^*$$

(where J is the momentum map of the action) [101], whose product with the dual pair $\mathfrak{g}^* \leftarrow\!\shortmid 0 \rightarrow pt$ in $\mathfrak{P}$ equals $S/G \leftarrow S/\!\!/G \rightarrow pt$ (if we assume G connected). In other words, the Marsden–Weinstein quotient $S/\!\!/G$ [1,63] may be interpreted in terms of the category $\mathfrak{P}$ (see Section 11 below for the significance of this observation.)

8. The classical imprimitivity theorem

There is a complete classical analogue of Mackey's theory of imprimitivity for (Lie) group actions [32,46,108]. Firstly, the classical counterpart of a representation of a C^*-algebra on a Hilbert space is a so-called *realization* of a Poisson manifold P on a symplectic manifold S [101]; this is a *complete* Poisson map $S \stackrel{\rho}{\to} P$.[41] The appropriate symplectic notion of irreducibility is that

$$\{X_{\rho^* f}(x) \mid f \in C^\infty(P)\} = T_x S$$

for all $x \in S$ (where X_g is the Hamiltonian vector field of $g \in C^\infty(S)$); it is easy to show (cf. Thm. I.2.6.7 in [46]) that ρ is irreducible iff S is symplectomorphic to a covering space of a symplectic leaf of P (and ρ is the associated projection followed by injection). In particular, any Poisson manifold has at least one irreducible realization.[42]

Secondly, we provide the classical counterpart of Definition 1. It goes without saying that in the present context G is a Lie group and M a manifold, all actions being smooth by definition.

Definition 9. Given a G-action on M, a G-covariant realization of M (seen as a Poisson manifold with zero Poisson bracket) is a complete Poisson map $S \stackrel{\rho}{\to} M$, where S is a symplectic manifold equipped with a strongly Hamiltonian G-action,[43] and $L_x(\rho^* f) = \rho^* L_x(f)$ for all $f \in C^\infty(M)$.

The significance of this definition and its analogy to Definition 1 are quite obvious; instead of a representation $\pi : C_0(M) \to B(\mathcal{H})$ one now has a Lie algebra homomorphism $\rho^* : C^\infty(M) \to C^\infty(S)$. Its relationship to the material in the preceding section is as follows:

Proposition 5. [106] *When G is connected, a G-covariant realization of M may equivalently be defined as a realization $S \stackrel{\sigma}{\to} \mathfrak{g}^* \ltimes M$ (equipped with the semidirect product Poisson structure) whose associated $\mathfrak{g}$-action on S is integrable (i.e. to a G-action on S).*

The $\mathfrak{g}$-action on S in question is given by $X \mapsto X_{\sigma^* \tilde{X}}$, where $X \in \mathfrak{g}$ defines a linear function $\tilde{X} : \mathfrak{g}^* \to \mathbb{C}$ by evaluation (and consequently also defines a function on $\mathfrak{g}^* \times M$ that is constant on M, which we denote by the same symbol). Of course, given $S \stackrel{\rho}{\to} M$ as in

[41] Some authors speak of a realization in case that ρ is surjective, but not necessarily complete. The completeness of ρ means that the Hamiltonian vector field $X_{\rho^* f}$ on S has a complete flow for each $f \in C_c^\infty(P)$ (i.e. the flow is defined for all times). This condition turns out to be the classical counterpart of the requirement that $\pi(a)^* = \pi(a^*)$ for representation s of a C^*-algebra. The analogy between completeness of the flow of a vector field and self-adjointness of an operator is even more powerful in the setting of unbounded operators; for example, the Laplacian on a Riemannian manifold M is essentially self-adjoint on $C_c^\infty(M)$ when M is geodesically complete [1].

[42] The appropriate symplectic notion of faithfulness is simply that ρ be surjective; it was recently shown by Crainic and Fernandes [19] that a Poisson manifold admits a faithful realization iff it is integrable; cf. Definition 5. Along with their solution of this integrability problem [18], this is one of the deepest results in symplectic geometry to date.

[43] In the sense that the G-action has an equivariant momentum map $J : S \to \mathfrak{g}^*$ [1,63].

Definition 9, one defines $S \xrightarrow{\sigma} \mathfrak{g}^* \ltimes M$ by $\sigma = (J, \rho)$; the nontrivial part of the proposition lies in the completeness of σ, given the completeness of ρ.

One then has the following classical analogue of Mackey's imprimitivity theorem.

Theorem 4. [108] *Up to symplectomorphism, there is a bijective correspondence between G-covariant realizations $S \xrightarrow{\rho} G/H$ of G/H (with zero Poisson structure) and strongly Hamiltonian H-spaces S_ρ, as follows*:

- *Given S_ρ, the Marsden–Weinstein quotient (at zero) $S^\rho = (T^*G \times S_\rho)//H$ is a G-covariant realization of G/H.*[44]
- *Conversely, given $S \xrightarrow{\rho} G/H$ there exists a strongly Hamiltonian H-space S_ρ such that $S \cong S^\rho$.*

This correspondence preserves irreducibility.

When G is connected, this correspondence may be seen as being between realizations $S \xrightarrow{\sigma} \mathfrak{g}^* \ltimes (G/H)$ whose associated $\mathfrak{g}$-action on S is integrable, and realizations $S_\rho \xrightarrow{J_\rho} \mathfrak{h}^*$ whose associated $\mathfrak{h}$-action on S_ρ is integrable.

The original proof of this theorem was lengthy and difficult [46,108]. Fortunately, as in the quantum case, there exists a direct categorical argument, according to which at least the last part of Theorem 4 is a consequence of (9) as well. Namely, the following analogue of Theorem 3 holds:

Theorem 5. [48] *Weinstein's map $\Gamma \mapsto A^*(\Gamma)$ is functorial from $\mathfrak{G}_c$ to $\mathfrak{P}$.*

Recall that $\mathfrak{G}_c$ is the full subcategory of $\mathfrak{G}$ whose objects are s-connected and s-simply connected Lie groupoids, and that the category $\mathfrak{P}$ of Poisson manifolds and dual pairs has been defined in the previous section. For example, $G \times (G/H)$ is an object in $\mathfrak{G}_c$ iff G is connected and simply connected. Assume this to be the case for the moment. As already mentioned, the category $\mathfrak{P}$ has a feature analogous to the category $\mathfrak{C}$ of C^*-algebras, namely that two objects are isomorphic iff they are Morita equivalent Poisson manifolds in the sense of Xu [105]. Consequently, similar to Corollary 1 one has:

Corollary 2. [50] *Weinstein's map $\Gamma \mapsto A^*(\Gamma)$ preserves Morita equivalence, in the sense that if Γ and Δ are Morita equivalent s-connected and s-simply connected Lie groupoids, then $A^*(\Gamma)$ and $A^*(\Delta)$ are Morita equivalent Poisson manifolds in the sense of Xu.*

Thus the Morita equivalence (9) of Lie groupoids implies the Morita equivalence

$$\mathfrak{g}^* \ltimes (G/H) \sim \mathfrak{h}^* \tag{18}$$

[44] The G-action inherited from the G-action on T^*G is given by pullback of left-multiplication, and the map $S^\rho \to G/H$ is inherited from the natural map $T^*G \to G \to G/H$.

of Poisson manifolds. As for C^*-algebras (and algebras in general), if two Poisson manifolds P_1, P_2 are Morita equivalent, then they have equivalent categories of realizations, and the equivalence bimodule implementing this Morita equivalence comes with an explicit procedure that defines a realization of P_2 given one of P_1, and vice versa. This procedure is a certain generalization of symplectic reduction [32,46,105] (much as the corresponding Rieffel induction procedure for C^*-algebras is a generalization of Mackey induction). In the case at hand, viz. (18), this precisely gives the prescription stated in Theorem 4, proving its last part at least for simply connected G. If G fails to be simply connected, one passes to its universal cover $\tilde{G}$, and lets it act on G/H via the projection $\tilde{G} \to G$. Hence $G/H \cong \tilde{G}/\hat{H}$ for some $\hat{H} \subset \tilde{G}$; Lie theory gives $\hat{\mathfrak{g}} = \mathfrak{g}$ and $\hat{\mathfrak{h}} = \mathfrak{h}$. The conclusion (18) still follows, this time as a consequence of $\tilde{G} \ltimes (\tilde{G}/\hat{H}) \sim \hat{H}$ rather than of (9).

We state a rather satisfying classical analogue of Proposition 1, which is essentially a corollary to Theorem 4.

Proposition 6. [64] *The symplectic leaves of the semidirect Poisson structure on* $\mathfrak{g}^* \ltimes M$ *are classified by pairs* $(\mathcal{O}, \mathcal{O}')$, *where* $\mathcal{O}$ *is a* G*-orbit in* M, *and* $\mathcal{O}'$ *is a coadjoint orbit of the stabilizer of an arbitrary point in* $\mathcal{O}$.

If we call the stabilizer in question H, the symplectic leaf $L_{(\mathcal{O},\mathcal{O}')}$ corresponding to the pair $(\mathcal{O}, \mathcal{O}')$ is given by

$$L_{(\mathcal{O},\mathcal{O}')} = \{(\theta, q) \in \mathfrak{g}^* \times Q \mid q \in \mathcal{O},\ (-\mathrm{Co}(s(q)^{-1})\theta \restriction \mathfrak{h}^*) \in \mathcal{O}'\}, \tag{19}$$

where $s : \mathcal{O} \simeq G/H \to G$ is an arbitrary section of the canonical principal H-bundle G over G/H, and Co is the coadjoint action of G on $\mathfrak{g}^*$.

Furthermore, one has a classical counterpart of (11), namely an isomorphism

$$\mathfrak{g}^* \ltimes (G/H) \cong (T^*G)/H \tag{20}$$

of Poisson manifolds. This may be generalized from the principal H-bundle G to arbitrary principal H-bundles P, provided that P is connected and simply connected (this assumption was not necessary in the quantum case). In that case, we may apply Corollary 2 to find a Morita equivalence of Poisson manifolds

$$(T^*P)/H \sim \mathfrak{h}^*. \tag{21}$$

9. Deformation quantization

Largely due to the functoriality of Connes's map (8) and its classical counterpart (15), we have observed a striking analogy between the C^*-algebra $C^*(\Gamma)$ and the Poisson manifold $A^*(\Gamma)$ associated to a Lie groupoid Γ. Beyond an analogy, the classical object $A^*(\Gamma)$ turns out to be related to its quantum counterpart through deformation quantization in the C^*-algebraic setting proposed by Rieffel:

Definition 10. [89,90] A C^*-algebraic deformation quantization of a Poisson manifold P is a continuous field of C^*-algebras $(A, A_\hbar)_{\hbar\in[0,1]}$,[45] where $A_0 = C_0(P)$, with a Poisson algebra $\tilde{A}_0$ densely contained in $C_0(P)$ and a cross-section $Q : \tilde{A}_0 \to A$ of π_0, such that, in terms of $Q_\hbar = \pi_\hbar \circ Q$, for all $f, g \in \tilde{A}_0$ one has

$$\lim_{\hbar\to 0} \| \frac{i}{\hbar}[Q_\hbar(f), Q_\hbar(g)] - Q_\hbar(\{f, g\})\|_\hbar = 0. \tag{22}$$

This has turned out to be an fruitful definition of quantization (cf. [46]). In many interesting examples the fiber algebras are non-isomorphic even away from $\hbar = 0$ (cf. [89,90] and Footnote 53 below), but in the case at hand the situation is simpler.[46]

Theorem 6. [47,56,83][47] *For any Lie groupoid Γ, the field $A_0 = C_0(A^*(\Gamma))$, $A_\hbar = C^*(\Gamma)$ for $\hbar \neq 0$, and $A = C^*(\Gamma^T)$, the C^*-algebra of the tangent groupoid Γ^T of Γ,[48] defines a C^*-algebraic deformation quantization of $A^*(\Gamma)$.*[49]

We refer to the literature cited for the specification of $\tilde{A}_0$, as well as for the proof of (22). The proof of the remainder of the theorem actually covers a much more general situation, as follows [83].[50]

Definition 11. A field of Lie groupoids is a triple (G, X, p), with G a Lie groupoid, X a manifold, and $p : \mathsf{G} \to X$ a surjective submersion such that $p = p_0 \circ r = p_0 \circ s$, where $p_0 = p \restriction \mathsf{G}_0$.

It follows that each $\mathsf{G}_x = p^{-1}(x)$ is a Lie subgroupoid of G over $\mathsf{G}_0 \cap p^{-1}(x)$, so that $\mathsf{G} = \coprod_{x\in X} \mathsf{G}_x$ as a groupoid. One may then form the convolution C^*-algebras $C^*(\mathsf{G})$ and $C^*(\mathsf{G}_x)$. Each $a \in C_c(\mathsf{G})$ (or $C_c^\infty(\mathsf{G})$) defines $a_x = a \restriction \mathsf{G}_x$ as an element of $C_c(\mathsf{G}_x)$ (etc.). These maps $C_c(\mathsf{G}) \to C_c(\mathsf{G}_x)$ are continuous in the appropriate norms, and extend to maps $\pi_x : C^*(\mathsf{G}) \to C^*(\mathsf{G}_x)$. Hence one obtains a field of C^*-algebras

$$(A = C^*(\mathsf{G}), A_x = C^*(\mathsf{G}_x))_{x\in X} \tag{23}$$

over X, where $a \in C^*(\mathsf{G})$ defines the section $x \mapsto \pi_x(a)$.[51] The question now arises when this field is continuous.

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[45] Here A is the C^*-algebra of sections of the given field, which defines its continuity structure. A continuous field $(A, A_x)_{x\in X}$ of C^*-algebras comes with surjective morphisms $\pi_x : A \to A_x$.

[46] Technically, the field in Theorem 6 is said to be trivial away from $\hbar = 0$, in the sense that $A_\hbar = B$ for all $\hbar \in (0, 1]$ and one has a short exact sequence $0 \to CB \to A \to A_0 \to 0$ (where $CB = C_0((0, 1], B)$ is the cone of B).

[47] See also [73] for a version of this result in the setting of formal deformation quantization (i.e. star products), and also cf. [82].

[48] Following Connes's definition of the special case of the pair groupoid $\Gamma = M \times M$ around 1980 (see [12]), the tangent groupoid (or adiabatic groupoid) of an arbitrary Lie groupoid was independently defined in [38,102]. See also [46,76].

[49] The same statement holds for the corresponding reduced groupoid C^*-algebras.

[50] This setting was originally suggested by Skandalis.

[51] A similar statement applies to the corresponding reduced C^*-algebras.

Lemma 1. [83] *The field* (23) *is continuous at all points where* G_x *is amenable* [3,85].[52]

For example, the tangent groupoid Γ^T of a given Lie groupoid Γ forms a field of Lie groupoids over $[0,1]$, with $\Gamma_0^T = A(\Gamma)$ (seen as a Lie *groupoid* instead of a Lie algebroid in the way every vector bundle $E \stackrel{\pi}{\to} M$ defines a Lie groupoid over its base space, namely by $s = t = \pi$ and fiberwise addition) and $\Gamma_\hbar^T = \Gamma$ for $\hbar \in (0,1]$. This eventually implies Theorem 6 (except for (22); the same strategy also leads to far-reaching generalizations thereof.[53]

In physics, Theorem 6 describes the quantization of particles with both internal and spatial degrees of freedom in a very wide setting. In noncommutative geometry, certain constructions of Connes in index theory turn out to be special cases of Theorem 6.[54] As to the ideology of noncommutative geometry, the theorem shows that the two fundamental classes of noncommutative manifolds, namely the ones defined by a singular quotient and the ones defined by deformation [12,13], overlap. For in case that the equivalence relation defining the quotient in question can be codified by a Lie groupoid Γ, the noncommutative space $C^*(\Gamma)$ associated with the quotient space is at the same time a deformation of the dual of its Lie algebroid.

Furthermore, Connes's philosophy in dealing with singular quotients, and especially his description of the Baum–Connes conjecture in Chapter II of [12], actually suggests a procedure for the quantization of such spaces. We explain this in a simple example [51]. Suppose a Lie group G acts on a manifold M; it acts on T^*M by pull-back, and we happen to be interested in quantizing the quotient $(T^*M)/G$. In case that the G-action is free and proper the situation is completely understood: the quotient is a Poisson manifold of the type $A^*(\Gamma)$ for $\Gamma = M \times_G M$, to which Theorem 6 applies (see also [46] for a detailed study of this case). However, if the G-action is not free (but still assumed to be proper), the quotient $(T^*M)/G$ may fail to be a manifold, let alone a Poisson manifold. According to Connes, one should replace the space $(T^*M)/G$ by the groupoid $T^*M \rtimes G$, and regard the associated noncommutative space $C^*(T^*M \rtimes G)$ as a *classical* space. If the G-action

[52] And similarly for the case of reduced C^*-algebras.

[53] Lemma 1 applies much more generally to fields of locally compact groupoids. In the context of C^*-algebraic deformation quantization, there are two typical situations. In the smooth (Lie) case studied in this paper, all $\mathsf{G}_\hbar$ are the same for $\hbar \neq 0$ but possibly not amenable, whereas G_0 is amenable. The former property then yields continuity at $\hbar = 0$ by the lemma, whereas the latter gives continuity on $(0,1]$. In the context of Definition 10, the reason why G_0 is amenable is that A_0 must be commutative, which implies that G_0 is a bundle of abelian groups. But such groupoids are always amenable [3]. In the étale case all $\mathsf{G}_\hbar$ are typically different from each other, but they are all amenable. See [9] for a description of noncommutative tori and the noncommutative four-spheres of Connes and Landi [14] (and of many other examples) as deformation quantizations along these lines.

[54] One instance is the map $p! : K^*(F^*) \to K_*(C^*(V,F))$ on p. 127 of [12], which plays a key role in the definition of the analytic assembly map for foliated manifolds. This is the K-theory map induced by the continuous field of Theorem 6, where Γ is the holonomy groupoid of the foliation. The index groupoid for a vector bundle map $L : E \to F$ defined in [12, Section II.6] is another example. Here one has a Lie groupoid $\Gamma = \mathrm{Ind}_L = F \rtimes_L E$ over F, whose Lie algebroid is $F \times_B E$. This is a vector bundle over B, and in the above formalism it should be regarded as a groupoid over F under addition in each fiber. Hence $A_0 = C^*(F \times_B E) \cong C_0(F \times E^*)$. The corresponding K-theory map occurs in Connes's construction of the Gysin map $f_! : K^*(X) \to K^*(Y)$ induced by a smooth map $f : X \to Y$ between manifolds.

is free, one has a Morita equivalence of Lie groupoids

$$T^*M \rtimes G \sim (T^*M)/G \tag{24}$$

which by Corollary 1 implies a Morita equivalence

$$C^*(T^*M \rtimes G) \sim C^*((T^*M)/G) \tag{25}$$

of C^*-algebras.[55] In general, we propose to quantize the singular space $(T^*M)/G$ by deforming $C^*(T^*M \rtimes G)$, which may be done by the field of Lie groupoids defined by the tangent groupoid Γ^T of $\Gamma = (M \times M) \rtimes G$. This field has fibers $\Gamma_0^T = TM \rtimes G$ (where TM is seen as a Lie groupoid, as explained above), and $\Gamma_\hbar^T = (M \times M) \rtimes G$. By Lemma 1 (which applies because $TM \rtimes G$ is amenable; see Lemma 2 in [51]), this field of groupoids leads to a continuous field of C^*-algebras with $A = C^*(\Gamma^T)$, etc., in the familiar way. The fibers of the latter field are simply $A_0 = C_0(T^*M) \rtimes G$ and $A_\hbar = K(L^2(M)) \rtimes G$ for all $\hbar \in (0, 1]$. To what extent this reflects physical desiderata remains to be seen.

10. Functorial quantization

The final application of groupoids to physics and noncommutative geometry we wish to describe in this paper is a functorial approach to quantization. In our opinion this forms the natural outcome of the categorical approach to Mackey's imprimitivity theorem described above. Beyond the desire to complete Mackey's program, why should one wish to turn quantization into a functor? Historically, quantum mechanics started with Heisenberg's paper *Über die quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen*[56][36]. One might argue that the proper mathematical reading of Heisenberg's idea of *Umdeutung* (*reinterpretation*) is that the transition from classical to quantum mechanics should be given by a functor. Indeed, attempts to make quantization functorial date back at least to van Hove's famous paper from 1951 [99] (see also [28,29]), the general conclusion being that functorial quantization is impossible (see [52] and refs. therein). However, all no-go theorems in this direction start from wrong and naive categories, both on the classical and on the quantum side.

Instead, though we have to warn the reader that we are presenting a program rather than a theorem here, it seems possible to interpret quantization as a functor $\mathcal{Q}$ from either the category $\mathfrak{S}$ (cf. Definition 2), or, more straightforwardly, from the category $\mathfrak{P}$ (see Definition 8; recall that $\mathfrak{P}$ is equivalent to a full subcategory of $\mathfrak{S}$) to the category $\mathfrak{KK}$ defined by Kasparov's bivariant K-theory (see [6,12]).[57] This was first proposed in [52–54]. Beyond the defining property of making quantization functorial, this program would:

[55] See [88] for the original, non-groupoid proof of (25).

[56] *On the quantum-theoretical reinterpretation of kinematical and mechanical relations*.

[57] The objects of $\mathfrak{KK}$ are separable C^*-algebras, and the arrows are $\mathrm{Hom}_{\mathfrak{KK}}(A, B) = KK(A, B)$, composed with Kasparov's product $KK(A, B) \times KK(B, C) \to KK(A, C)$.

- Unify deformation quantization and geometric quantization into a single operation (the former becoming the object side of the quantization functor and the latter the arrow side);
- Imply the functoriality of shriek maps in K-theory [4], in particular providing a natural home for Connes-style proofs and generalizations of index theorems [12,15];
- Imply the "quantization commutes with reduction" conjecture of Guillemin and Sternberg [31];
- Provide unlimited generalizations of this conjecture, e.g., to noncompact Lie groups and Lie groupoids (see [39] for the former).

It should be clear that the use of groupoids is essential in this program, since the classical category $\mathfrak{S}$ of symplectic groupoids and principal symplectic bibundles either forms the domain of the quantization functor $\mathcal{Q}$, or, in case one more naturally starts from $\mathfrak{P}$, plays an essential role in the definition of the latter category.

Let us indeed construe quantization as a functor $\mathcal{Q} : \mathfrak{P} \to KK$. This means that quantization sends (isomorphism classes of) dual pairs into (homotopy classes of) Kasparov bimodules. More precisely, if Poisson manifolds P_1 and P_2 are quantized by (separable) C^*-algebras $\mathcal{Q}(P_1)$ and $\mathcal{Q}(P_2)$, respectively, a dual pair $P_1 \leftarrow M \rightarrow P_2$ should be quantized by an element

$$\mathcal{Q}(P_1 \leftarrow M \rightarrow P_2) \in KK(\mathcal{Q}(P_1), \mathcal{Q}(P_2)), \tag{26}$$

where $KK(-,-)$ is the usual Kasparov group [6,12]. Roughly speaking, the construction of $\mathcal{Q}(P)$ should be done by some C^*-algebraic version of deformation quantization, whereas that of $\mathcal{Q}(P_1 \leftarrow M \rightarrow P_2)$ should come from a far-reaching generalization of geometric quantization first proposed, in special cases, by Raoul Bott;[58] see [33,94]. This proposal turns out to be closely related to Connes's construction of shriek maps [12,15].

To explain the construction of (26), we assume that the symplectic manifold (M, ω) is prequantizable. Cf. [33,75] for details of the following approach to geometric quantization. One picks an almost complex structure $\mathcal{J}$ on M that is compatible with ω (in that $\omega(-, \mathcal{J}-)$ is positive definite and symmetric). This $\mathcal{J}$ canonically induces a Spin^c structure on TM, which should subsequently be twisted by a prequantization line bundle L line bundle over M to obtain a Spin^c structure $(P, \cong)$ on M.[59] Denote the (complex) spin representation of Spin$^c(n)$ on the finite-dimensional Hilbert space S_n by Δ_n. One may then form the associated spinor bundle $\mathcal{S}_n = P \times_{\Delta_n} S_n$, with Dirac operator $\not{D} : C^\infty(M, \mathcal{S}_n) \to C^\infty(M, \mathcal{S}_n)$. For even n (the case that applies here, as M is symplectic) the spin representation decomposes into two irreducibles $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$ on $S_n = S_n^+ \oplus S_n^-$, so that also the vector bundle $\mathcal{S}_n$ decomposes accordingly as $\mathcal{S}_n = \mathcal{S}_n^+ \oplus \mathcal{S}_n^-$. Being odd with respect to this decomposition, the Dirac operator then splits accordingly as $\not{D}^\pm = C^\infty(M, \mathcal{S}^\pm) \to C^\infty(M, \mathcal{S}^\mp)$.

[58] This was done in seminars and conversations; no paper by Bott containing his proposal seems to exist (V. Guillemin, R. Sjamaar, private communications).

[59] We here define a Spin^c structure on M as an equivalence class of principal Spin$^c(n)$-bundle P over M with an isomorphism $P \times_\pi \mathbb{R}^n \cong TM$ of vector bundles. Here $n = \dim(M)$ and the bundle on the left-hand side is the bundle associated to P by the defining representation of $SO(n)$. Connes's construction of shriek maps lacks the twisting with the prequantization line bundle.

Given a dual pair $P_1 \leftarrow M \rightarrow P_2$, the fundamental idea is to use the map $M \rightarrow P_2$ to turn the appropriate completion of $C_c^\infty(M, \mathcal{S}_n)$ to a graded Hilbert $C^*(\mathcal{Q}(P_2))$ module $\mathcal{E}$, and subsequently, to use the map $P_1 \leftarrow M$ to construct an action of $C^*(\mathcal{Q}(P_1))$ on $\mathcal{E}$, producing a $C^*(\mathcal{Q}(P_1))$-$C^*(\mathcal{Q}(P_2))$ graded Hilbert bimodule. The final step is to employ the Dirac operator $\not{D}$ to enrich this bimodule into a Kasparov cycle, whose homotopy class defines the element (26) we are after.

This procedure has so far been carried through in a few cases only, namely those in which Theorem 6 states how the Poisson manifolds P_i are to be quantized, and in which simultaneously techniques from the literature on the Baum–Connes conjecture [5,12,98] are available to construct (26) according to the procedure just sketched. The simplest case is $P_1 = P_2 = pt$ (i.e. a point) and M an arbitrary compact prequantizable symplectic manifold.[60] Most people would agree that $\mathcal{Q}(pt) = \mathbb{C}$, and under the isomorphism $KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$ the Kasparov cycle defined by $\not{D}$ is just the Fredholm index of $\not{D}_+$ [6]. This number, then, is Bott's quantization of (M, ω). Consequently, we have

$$\mathcal{Q}(pt \leftarrow M \rightarrow pt) = \mathrm{Index}(\not{D}_+). \tag{27}$$

11. Quantization commutes with reduction

The above definition of quantization gains in substance when one passes to a dual pair $M/G \leftarrow M \rightarrow \mathfrak{g}^*$ defined by a strongly Hamiltonian group action $G \circlearrowright M$ in the usual way [101]. For simplicity, we will actually use the dual pair $pt \leftarrow M \rightarrow \mathfrak{g}^*$.[61] Theorem 6 tells us that $\mathcal{Q}(\mathfrak{g}^*) = C^*(G)$, where G is any Lie group with Lie algebra G; we take the connected and simply connected one.[62] Hence the quantization of the dual pair $pt \leftarrow M \rightarrow \mathfrak{g}^*$ should be an element of the Kasparov group $KK(\mathbb{C}, C^*(G))$.

This element can be defined when the G-action is proper and cocompact (i.e. M/G is compact), and lifts to an action on the principal bundle P defining the Spin^c structure. Namely, in that case one regards $\not{D}$ as an operator on the graded Hilbert space $L^2(M, \mathcal{S}_n)$ of L^2-sections of $\mathcal{S}_n$, which at the same time carries a natural representation π of $C_0(M)$ by multiplication operators, as well as a natural unitary representation $U(G)$. Provided that in addition the Dirac operator $\not{D}$ is almost G-invariant in the sense that $[U(x), \not{D}]$ is bounded for each $x \in G$, these data specify an element $[L^2(M, \mathcal{S}_n), \pi(C_0(M)), U(G), \not{D}]$ of the equivariant analytic K-homology group $K_0^G(M) = KK^G(C_0(M), \mathbb{C})$ [37]. Here we suppress the grading of the Hilbert space in question in our notation. Let

$$\mathrm{Index}_G : K_0^G(M) \rightarrow K_0(C^*(G))$$

[60] Let us note that the associated dual pair $pt \leftarrow M \rightarrow pt$ does not define an element of our category $\mathfrak{P}$, but this nuisance does not stop us from proceeding.

[61] This dual pair does not define an element of $\mathfrak{P}$, but this does not affect any of our arguments.

[62] Here the use of the category $\mathfrak{S}$ as the domain of the quantization functor $\mathcal{Q}$ is more satisfactory. The classical data is then formed by the G-action on M itself (in the guise of the associated symplectic action of the symplectic groupoid T^*G), instead of the associated momentum map $M \rightarrow \mathfrak{g}^*$. This refinement is, of course, essential when G is discrete.

be the analytic assembly map as defined by Baum et al. [5], seen however as a map taking values in $K_0(C^*(G))$ instead of $K_0(C_r^*(G))$ (cf. [98] for this point). For simplicity we write

$$\mathrm{Index}_G(\not{D}_+) \equiv \mathrm{Index}_G([L^2(M, \mathcal{S}_n), \pi(C_0(M)), U(G), \not{D}]). \tag{28}$$

We then define the quantization of the dual pair $pt \leftarrow M \rightarrow \mathfrak{g}^*$ as

$$\mathcal{Q}(pt \leftarrow M \rightarrow \mathfrak{g}^*) = \alpha_{C^*(G)}(\mathrm{Index}_G(\not{D}_+)), \tag{29}$$

where $\alpha_A : K_0(A) \rightarrow KK(\mathbb{C}, A)$ is the natural isomorphism one has for any separable C^*-algebra A [6]. As required, (29) defines an element of

$$KK(\mathcal{Q}(pt), \mathcal{Q}(\mathfrak{g}^*)) = KK(\mathbb{C}, C^*(G)).$$

For a much simpler example, whose significance will become clear shortly, consider the dual pair $\mathfrak{g}^*_- \hookleftarrow 0 \rightarrow pt$, where 0 (seen as a coadjoint orbit of G) is the zero element of the vector space $\mathfrak{g}^*$, equipped with minus the Lie–Poisson structure. Its quantization should be an element of the Kasparov representation ring $KK(C^*(G), \mathbb{C})$, which we simply take to be the graded Hilbert space $\mathcal{H} = \mathbb{C} \oplus 0$ carrying the trivial representation of G, with $F = 0$. We denote this element by $[\mathbb{C}, 0, 0]$, so that

$$\mathcal{Q}(\mathfrak{g}^* \hookleftarrow 0 \rightarrow pt) = [\mathbb{C}, 0, 0]. \tag{30}$$

Let

$$\tau_* : KK(\mathbb{C}, C^*(G)) \rightarrow KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$$

be the map functorially induced by the morphism $\tau : C^*(G) \rightarrow \mathbb{C}$ given by the trivial representation of G.[63] Functoriality of the Kasparov product

$$KK(\mathbb{C}, C^*(G)) \times KK(C^*(G)), \mathbb{C} \rightarrow KK(\mathbb{C}, \mathbb{C}) \stackrel{\rightarrow}{\cong} \mathbb{Z}$$

then yields

$$y \times [\mathbb{C}, 0, 0] = \tau_*(y) \tag{31}$$

for any $y \in KK(\mathbb{C}, C^*(G))$. In particular, (29) and (30) give

$$\mathcal{Q}(pt \leftarrow M \rightarrow \mathfrak{g}^*) \times \mathcal{Q}(\mathfrak{g}^*_- \hookleftarrow 0 \rightarrow pt) = \tau_*(\mathrm{Index}_G(\not{D}^M_+)); \tag{32}$$

to avoid confusion later on, we have added a suffix M to the pertinent Dirac operator.

On the classical side, in the category $\mathfrak{P}$ we compute

$$(pt \leftarrow M \rightarrow \mathfrak{g}^*) \circ (\mathfrak{g}^*_- \hookleftarrow 0 \rightarrow pt) = pt \leftarrow M/\!/G \rightarrow pt, \tag{33}$$

[63] For $f \in C_c(G)$ one simple has $\tau(f) = \int_G \mathrm{d}x\, f(x)$. This is the reason why we use $C^*(G)$ rather than $C_r^*(G)$, as is customary in the Baum–Connes conjecture: for τ is not continuous on $C_r^*(G)$ (unless G is amenable).

where $M /\!\!/ G$ is the Marsden–Weinstein quotient. Assuming that $M /\!\!/ G$ is prequantizable (this is a theorem in the compact case [33]), we have already seen from (27) that

$$\mathcal{Q}(pt \leftarrow M//G \rightarrow pt) = \mathrm{Index}(\not{D}_+^{M//G}), \tag{34}$$

where we have denoted the appropriate Dirac operator on $M//G$ by $\not{D}^{M//G}$.

Functoriality of quantization would imply

$$\mathcal{Q}(pt \leftarrow M \rightarrow \mathfrak{g}^*) \times \mathcal{Q}(\mathfrak{g}^*_- \hookleftarrow 0 \rightarrow pt) = \mathcal{Q}((pt \leftarrow M \rightarrow \mathfrak{g}^*) \circ (\mathfrak{g}^*_- \hookleftarrow 0 \rightarrow pt)). \tag{35}$$

Using (32) and (33), this amounts to

$$\tau_*(\mathrm{Index}_G(\not{D}_+^M)) = \mathrm{Index}(\not{D}_+^{M//G}). \tag{36}$$

For G and M compact, this is precisely the so-called Guillemin–Sternberg conjecture that "quantization commutes with reduction" [31] in its modern form [33,65,94].[64] To see this, note that for M compact the Dirac operator $\not{D}_+$ is Fredholm, whereas for G compact one has $K_0(C^*(G)) \cong R(G)$, the representation ring of G. Consequently, $\mathrm{Index}_G(\not{D}_+)$ defines an element of $R(G)$, and the map $\tau_* : R(G) \rightarrow R(e) \cong \mathbb{Z}$ is just $[V] - [W] \mapsto \dim(V_0) - \dim(W_0)$, where $V_0 \subset V$ is the space of G-invariant vectors, etc.

For G countable (acting properly and cocompactly on M, as stated before), (36) boils down to the naturality of the Baum–Connes assembly map for countable discrete groups [98]. Combining this fact with the validity of (36) for compact G and M, it can be shown that (36) holds for any strongly Hamiltonian proper cocompact action of G on a possibly noncompact symplectic manifold, provided that G contains a discrete normal subgroup Γ with G/Γ compact [39].

Let us close this paper in the right groupoid spirit by pointing out that all arguments in this section should be carried out for Lie groupoids instead of Lie groups. For example, the pertinent symplectic reduction procedure (generalizing Marsden–Weinstein reduction) was first studied in [66], and can be reinterpreted in terms of the product in the category $\mathfrak{P}$ just as in the group case. A very interesting special case comes from foliation theory, as follows (cf. [11,12,15,38]). Let (V_i, F_i), $i = 1, 2$, be foliations with associated holonomy groupoids $G(V_i, F_i)$ (assumed to be Hausdorff for simplicity). A smooth generalized map f between the leaf spaces V_1/F_1 and V_2/F_2 is defined as a principal bibundle M_f between the Lie groupoids $G(V_1, F_1)$ and $G(V_2, F_2)$. Classically, such a bibundle defines a dual pair $T^*F_1 \leftarrow T^*M_f \rightarrow T^*F_2$ [50]. Here $TF_i \subset TV_i$ is the tangent bundle to the foliation (V_i, F_i), whose dual bundle T^*F_i has a canonical Poisson structure.[65] Quantum mechanically, f defines an element [11,38]

$$f_! \in KK(C^*(G(V_1, F_1)), C^*(G(V_2, F_2))).$$

In the functorial approach to quantization, $f_!$ is interpreted as the quantization of the dual pair $T^*F_1 \leftarrow T^*M_f \rightarrow T^*F_2$. The functoriality of quantization among dual pairs of the

[64] This conjecture is, in fact, a theorem [41,65,75,97], but the name "conjecture" is still generally used.

[65] The best way to see this is to interpret TF_i as the Lie algebroid of $G(V_i, F_i)$.

same type should then follow from the computations in [38] on the quantum side and [50] on the classical side. The construction and functoriality of shriek maps in [4,11] is a special case of this, in which the V_i are both trivially foliated.

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